SHARP ASYMPTOTICS OF THE QUASIMOMENTUM

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Abstract. We consider the Schrödinger operator with a periodic potential p on the real line. We assume that p belongs to the Sobolev space $\mathscr{H}_m$ on the circle for some $m \geqslant -1$, and we determine the asymptotics of the quasimomentum and the Titchmarsh-Weyl functions, the Bloch functions at high energy.

1. Introduction and main results

Consider the Schrödinger operator H acting in the Hilbert space $L^2(\mathbb{R})$ and given by

$$Hf = -f'' + pf.$$

Here the potential p is 1-periodic and belongs to the Sobolev space $\mathscr{H}_m$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$p \in \mathscr{H}_m = \{p^{(m)} \in L^2(\mathbb{T})\}, \qquad m \geqslant -1. \tag{1.1}$$

We recall the results from [K1] about the operator H. The spectrum of H is absolutely continuous and has the form $\sigma(H_0) = \bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$, where the bands $\mathfrak{S}_n$ and gaps γ_n are given by

$$\mathfrak{S}_n = [E_{n-1}^+, E_n^-], \qquad \gamma_n = (E_n^-, E_n^+), \quad \forall n \in \mathbb{N} = \{n : n = 1, 2, 3, ...\},$$

see Fig. 1. Without loss of generality, we may assume $E_0^+ = 0$. Here the $E_n^\pm$ satisfy

$$0 = E_0^+ < E_1^- \leqslant E_1^+ \cdots \leqslant E_{n-1}^+ < E_n^- \leqslant E_n^+ < \dots \tag{1.2}$$

If $p \in \mathscr{H}_m$, then it is known that there are infinitely many non-degenerate gaps, i.e. $E_n^- < E_n^+$, unless p is arbitrarily often differentiable, and all gaps are non-degenerate generically (see e.g. [MO], [K1]). The sequence (1.2) is the spectrum of the equation

$$-y'' + py = \lambda y, \tag{1.3}$$

with the condition of 2-periodicity, $y(x+2) = y(x)$ $(x \in \mathbb{R})$. If a gap degenerates, $\gamma_n = \emptyset$ for some n, then the corresponding bands $\mathfrak{S}_n$ and $\mathfrak{S}_{n+1}$ touch. This happens when $E_n^- = E_n^+$; this number is then a double eigenvalue of the 2-periodic problem (1.3). The lowest eigenvalue $E_0^+ = 0$ is always simple and has a 1-periodic eigenfunction. Generally, the eigenfunctions corresponding to the eigenvalues $E_{2n}^\pm$ are 1-periodic, and those for $E_{2n+1}^\pm$ are 1-anti-periodic in the sense that $y(x+1) = -y(x)$ $(x \in \mathbb{R})$.

In the case of the potential $p \in \mathscr{H}_m, m \geqslant 0$, throughout the paper, we shall denote by $\vartheta(x,z), \varphi(x,z)$ the two solutions forming the canonical fundamental system of the unperturbed equation

$$-y'' + py = z^2 y, \tag{1.4}$$

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under the initial conditions

$$\varphi'(0,z) = \vartheta(0,z) = 1, \qquad \varphi(0,z) = \vartheta'(0,z) = 0.$$

Here and in the following " $'$ " denotes the derivative w.r.t. the first variable. In the following, we shall treat the momentum $z = \sqrt{\lambda}$ (as opposed to the energy $z^2 = \lambda$) as the principal spectral variable. The Lyapunov function (which is the Hill discriminant for $m \geqslant 0$) of the periodic equation is then defined by

$$\Delta(z) = \frac{1}{2}(\varphi'(1,z) + \vartheta(1,z)).$$

In the case $m = -1$ we denote the Lyapunov function also by $\Delta(z)$. In the last case the definition of $\Delta(z)$ is more complicated and is given in Section 4. Recall that the function $\Delta(z)$ is entire and even $\Delta(-z) = \Delta(z), z \in \mathcal{Z}$.

We introduce the *quasimomentum* $k(\cdot)$ for H as $k(z) = \arccos \Delta(z), z \in \mathcal{Z}$, where $\mathcal{Z}$ is the cut domain (see Fig. 1 and 2) given by

$$\mathcal{Z} = \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \overline{g}_n, \quad \text{where} \quad g_n = (e_n^-, e_n^+) = -g_{-n}, \quad e_n^\pm = \sqrt{E_n^\pm} > 0, \quad n \geqslant 1, \quad g_0 = \emptyset. \quad (1.5)$$

Note that $\Delta(e_n^\pm) = (-1)^n$ and if $\lambda \in \gamma_n, n \geqslant 1$, then $z \in g_{\pm n}$, and if $\lambda \in \gamma_0 = (-\infty, E_0^+)$, then $z \in i\mathbb{R}_\pm$. The function k is analytic in $\mathcal{Z}$ and satisfies

$$\begin{aligned} &(i) && k(z) = z + o(1) \quad \text{as} \quad \operatorname{Im} z \to \infty, \\ &(ii) && k(0) = 0, \qquad \operatorname{Re} k(z \pm i0)|_{[e_n^-, e_n^+]} = \pi n \quad (n \in \mathbb{Z}), \\ &(iii) && k(-z) = -k(z), \qquad \forall \quad z \in \mathcal{Z}, \\ &(iv) && \pm \operatorname{Im} k(z) > 0, \qquad \forall\, z \in \mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}, \end{aligned} \quad (1.6)$$

see ([MO], [KK]). Moreover, k is a conformal mapping from $\mathcal{Z}$ onto the quasimomentum domain $\mathcal{K}$ given by

$$\mathcal{K} = \mathbb{C} \setminus \cup \overline{\Gamma}_n, \qquad \Gamma_n = (\pi n - i h_n, \pi n + i h_n), \quad (1.7)$$

see Figs. 2 and 3. Here Γ_n is a vertical cut of the height $h_n = h_{-n} \geqslant 0, h_0 = 0$. The height h_n is determined by the equation $\cosh h_n = |\Delta(e_n)| \geqslant 1$, where $e_n \in [e_n^-, e_n^+]$ is such that $\Delta'(e_n) = 0$. Note that the point e_n is unique for each $n \in \mathbb{Z}$. The function k maps the cut g_n onto the cut Γ_n.

We have obtained a conformal mapping $k : \mathcal{Z} \to \mathcal{K}$, called *the quasimomentum mapping* (or shortly the quasimomentum), which generalizes the classical quasimomentum (see e.g. [RS]). A point $z \in \mathcal{Z}$ is called *a momentum* and a point $k \in \mathcal{K}$ is called *a quasimomentum*. The abstract quasimomentum, which we have just defined is related to the spectral theory of the Hill operator H by the following construction invented in [F1], [F], [MO] for the L^2 potentials and generalized in [K1] for the potential from $\mathscr{H}_{-1}$. Some asymptotics of the quasimomentum for $p \in \mathscr{H}_0$ were obtained in [F2], outside some neighborhoods of gaps. The quasimomentum for the Schrödinger operator $-\frac{d^2}{dx^2} + V$ acting on the real line where V is a periodic $N \times N$ matrix-valued potential was studied in [CK]. We would like to add that the properties of the quasimomentum are important in many different fields, see e.g. : inverse problem [F], [GT], [KK1], [K1], [MO], non-linear equations [C], [GWH], and so on.

For any $p \in \mathscr{H}_m, m \geqslant 0$ we define the integrals

$$P_{-1} = \frac{\int_0^1 p dx}{2}, \quad P_0 = \frac{\int_0^1 p^2 dx}{2^3}, \quad P_j = \frac{\|p^{(j)}\|^2 + \int_0^1 F_j dx}{2^{3+2j}}, \quad j = 1, ..., m. \quad (1.8)$$

Here F_j is some polynomial of $p, p', p'', \dots, p^{(j-1)}$. In particular, we have

$$F_1 = 2p^3, \qquad F_2 = 10pp'^2 + 5p^4, \qquad F_3 = 14pp''^2 + 70p^2p'^2 + 112p^5, \dots, \tag{1.9}$$

see [MM], [MO], where all $P_j > 0$ if $p \neq 0$ and $E_0^+ = 0$, since we have (2.9). Introduce the functions

$$K_m(z) = \frac{P_{-1}}{z} + \frac{P_0}{z^3} + ... + \frac{P_{m-1}}{z^{2m+1}}, \tag{1.10}$$

and define the domains

$$\mathcal{Z}_\varepsilon = \{z \in \mathcal{Z} : \operatorname{dist}\{z, g\} > \varepsilon\}, \quad \varepsilon > 0, \qquad \text{where} \quad g = \bigcup_{n \in \mathbb{Z}} g_n.$$

Theorem 1.1. *Let $p \in \mathscr{H}_m$ for some $m \geqslant 0$ and let $A, \varepsilon > 0$. Then*

$$k = z - K_m(z) + f_{m+1}(z), \quad f_{m+1}(z) = \frac{1}{\pi z^{2m+2}} \int_{\mathbb{R}} \frac{t^{2m+2} v(t) dt}{t - z}, \qquad z \in \mathcal{Z}, \tag{1.11}$$

where f_{m+1} has the following asymptotics as $|z| \to \infty$:

$$f_{m+1}(z) = -\frac{P_m + o(1)}{z^{2m+3}} \qquad as \qquad z \in \{z = x + iy \in \mathbb{C} : y > A|x|\}, \tag{1.12}$$

$$f_{m+1}(z) = \frac{O(1)}{z^{2m+2}} \quad \text{as} \quad z \in \mathcal{Z}_\varepsilon, \tag{1.13}$$

$$|f_{m+1}(z)| \leqslant \frac{|\gamma_n|}{2\pi n} + b_n, \qquad b_n = \frac{O(|\gamma_n|)}{n^3} + \frac{O(1)}{n^{2m+2}} \qquad \operatorname{dist}\{z, g_n\} \leqslant \varepsilon. \tag{1.14}$$

Moreover, the asymptotic estimate (1.14) is sharp, since

$$f_{m+1}(e_n^\pm) = \mp \frac{|\gamma_n|}{2\pi n}(1 + o(1)) \quad \text{as} \quad n \to \infty. \tag{1.15}$$

Remarks. 1) Recall that $p \in \mathscr{H}_m$ if and only if $(n^m|\gamma_n|)_1^\infty \in \ell^2$, see [MO], [K3].

2) (1.12)-(1.14) give 3 types of asymptotics. The "best" asymptotics (1.12) has the form $f_{m+1}(z) = \frac{O(1)}{z^{2m+3}}$ and the "bad" asymptotics (1.14) has the form $f_{m+1}(z) = \frac{O(n^m|\gamma_n|)}{n^{m+1}}$. There is a big difference between the sharp asymptotics (1.12) and (1.14), since due to (1.15) the asymptotics (1.14) is sharp.

3) Shenk and Shubin [SS] determined complete asymptotic expansions of the integrated density of states for $p \in C^\infty(\mathbb{R})$. Recall that the integrated density of states is given by $\frac{1}{\pi} \operatorname{Re} k(z), z \in \mathbb{R}$.

4) The asymptotics (1.12)-(1.14) give an asymptotics of the integrated density of states for the case $p \in \mathscr{H}_m$. If $p \in C^\infty(\mathbb{R})$, then the theorem gives complete asymptotic expansions of the quasimomentum $k(z)$.

5) The complete asymptotic expansion of the integrated density of states of multidimensional almost-periodic Schrodinger operators were determined by Parnovski, Shterenberg [PS1], see also [KP], [PS2] and the references therein.

In Section 4 we consider the case of distributional potentials $p \in \mathscr{H}_{-1}$.

In order to write the more complete results about the asymptotics for the Hill operator, we determine the asymptotics of the Bloch functions and the Titchmarsh-Weyl function. Note that, although asymptotic expressions for the Bloch functions and the Titchmarsh-Weyl function for the case $p \in \mathscr{H}_m$ have not been formally written out anywhere previously, this

result can be regarded as known, since it can easily be obtained with the help of the results of Marchenko and Ostrovskii [MO].

We introduce the Bloch functions $\Psi_\pm$ of H defined by (see [T])

$$\Psi_\pm(x,z)=\varphi(x,z)+M_\pm(z)\vartheta(x,z),\qquad (x,z)\in[0,1]\times\mathcal{Z},\tag{1.16}$$

where $M_\pm(z)$ is the Titchmarsh-Weyl function given by

$$M_\pm(z)=\frac{\beta(z)\pm\sin k(z)}{\varphi(1,z)},\qquad \beta(z)=\frac{\varphi'(1,z)-\vartheta(1,z)}{2}.\tag{1.17}$$

Furthermore, we introduce the model function (see Lemma 3.1 in[MO])

$$\xi_m(x,z)=zx-i\int_0^x\sum_1^m\frac{\varkappa_j(t)}{(2iz)^j}dt,\qquad (x,z)\in[0,1]\times\mathbb{C},\quad z\neq 0.\tag{1.18}$$

Here the functions $\varkappa_j$ are constructed with the help of the recursion relations:

$$\varkappa_{j+1}=-\varkappa_j'-\sum_1^{j-1}\varkappa_{j-s}\varkappa_s,\quad j=1,2,...,m-1,\tag{1.19}$$

where, in particular,

$$\begin{aligned}\varkappa_1=p,\quad \varkappa_2=-p',\quad \varkappa_3=p''-p^2,\quad \varkappa_4=-p'''+4pp',\quad ....\\ \varkappa_j=(-1)^{j-1}p^{(j-1)}+\mathcal{P}_{j-3},\quad j=2,3,...,m,\end{aligned}\tag{1.20}$$

and $\mathcal{P}_j$ is a polynomial in $p,p',p'',...,p^{(j)}$.

Theorem 1.2. *Let $p\in\mathscr{H}_m$ for some $m\geqslant 0$ and let $\varepsilon>0, r\geqslant 1$. Assume that E_0^+ is any real number. Then the following asymptotics hold true as $|z|\to\infty$:*

$$M_\pm(z)=i\xi_m'(0,\pm z)+O(z^{1-m}),\tag{1.21}$$

$$\Psi_\pm(x,z)=e^{i\xi_m(x,\pm z)}+O(z^{-m}),\tag{1.22}$$

as $z\in\mathcal{Z}_\varepsilon, |\operatorname{Im} z|<r$, uniformly in $x\in[0,1]$.

Moreover, if in addition $E_0^+=0$, then

$$\begin{gathered}k(z)=\xi(1,z)+O(z^{-m}),\\ \frac{(-1)^j}{2^{2j+1}}\int_0^1\varkappa_{2j+1}(t)dt=Q_{2j},\qquad \int_0^1\varkappa_{2j}(t)dt=0,\quad j\geqslant 0,\end{gathered}\tag{1.23}$$

as $|\operatorname{Im} z|\leqslant r$ and $|z|\to\infty$.

Remarks 1) Shenk and Shubin [SS] determined complete asymptotic expansions of the Bloch functions for $p\in C^\infty(\mathbb{R})$. There are some asymptotics of the Bloch functions for $p\in L^1(0,1)$ in [T], [F], [F2].

2) In the proof of the theorem we use the standard asymptotics of the solutions of the equation $-y''+p(x)y=z^2y$ for large z from [MO].

2. Asymptotics of the quasimomentum

Recall that the quasimomentum $k(z)$ is a conformal mapping from the momentum domain $\mathcal{Z}$ onto the quasi-momentum domain $\mathcal{K}$ given by (see Fig. 2 and 3)

$$\begin{aligned} \mathcal{Z} = \mathbb{C} \setminus \cup \overline{g}_n, \quad \text{where} \quad g_n = (e_n^-, e_n^+) = -g_{-n}, \quad e_n^\pm = \sqrt{E_n^\pm} > 0, \quad n \geqslant 1, \quad g_0 = \emptyset, \\ \mathcal{K} = \mathbb{C} \setminus \cup \overline{\Gamma}_n, \quad \text{where} \quad \Gamma_n = (\pi n + i h_n, \pi n - i h_n), \quad h_n = h_{-n} \geqslant 0, \quad n \geqslant 1, \quad h_0 = 0. \end{aligned} \tag{2.1}$$

The height h_n is determined by the equation $\cosh h_n = |\Delta(e_n)| \geqslant 1$, where $e_n \in [e_n^-, e_n^+]$ is such that $\Delta'(e_n) = 0$. Note that e_n is unique for each $n \in \mathbb{Z}$. Cutting the n-th momentum gap g_n (if non-empty), we obtain a cut g_n^c with upper rim g_n^+ and lower rim g_n^-. Below, we will identify this cut g_n^c and the union of the upper rim (gap) $\overline{g}_n^+$ and the lower rim (gap) $\overline{g}_n^{\,-}$, i.e.,

$$g_n^c = \overline{g}_n^+ \cup \overline{g}_n^-, \qquad \text{where } g_n^\pm = g_n \pm i0; \quad \text{and } z \in g_n \Rightarrow z \pm i0 \in g_n^\pm. \tag{2.2}$$

Any non-degenerate (degenerate) cut Γ_n is connected in the some way with the non-degenerate (degenerate) gap γ_n and the momentum gap g_n. We introduce the decomposition $k = u + iv$, where u, v are real harmonic functions in $\mathcal{Z}$. The function $u(z) = \operatorname{Re} k(z)$ is strongly increasing on each band σ_n and equals πn on each gap $[z_n^-, z_n^+]$, $n \in \mathbb{Z}$; the function $v(z) = \operatorname{Im} k(z)$ equals zero on each band σ_n, is strongly concave on each gap g_n and has the maximum h_n in g_n, attained at some point e_n, so that $h_n = v(e_n)$. Here and below we write

$$v(z) = v(z + i0) \qquad \text{as} \quad z \in \mathbb{R}. \tag{2.3}$$

If $h_n = 0$, then n-the gap is empty and $e_n^- = e_n^+ = e_n$. These and others properties of the comb mappings can be found in [KK],[MO].

Introduce the real spaces

$$\ell^a = \Big\{ f = (f_n)_{n \geqslant 1}, \quad \|f\|_a < \infty \Big\}, \qquad \|f\|_a^a = \sum_{n \geqslant 1} |f_n|^a < \infty, \ a \geqslant 1.$$

Now we briefly discuss the properties of the general quasimomentum mapping $k = u + iv$, as a function of $z = x + iy \in \mathcal{Z}$. Their proof may be found in [MO], [K1], [K2], [K4].

1) $v(z) \geqslant \operatorname{Im} z > 0$ and $v(z) = -v(\overline{z})$ for all $z \in \mathbb{C}_+$ and

$$k(-z) = -k(z) = \overline{k}(\overline{z}), \quad \text{all } z \in \mathcal{Z}. \tag{2.4}$$

2) $v(z) = 0$ for all $z \in \sigma_n = [e_{n-1}^+, e_n^-], n \in \mathbb{Z}$.

3) If some $g_n \neq \emptyset, n \in \mathbb{Z}$, then $v(z) > 0$ and $v''(z) < 0$ for all $z \in g_n$, and $v(z)$ has a maximum at $e_n \in g_n$ such that $v'(e_n) = 0$, see Fig. 3, and $\Delta'(e_n) = 0$ and

$$v(z + i0) = -v(z - i0) > 0, \qquad \text{all } z \in g_n \neq \emptyset, \tag{2.5}$$

$$|g_n| \leqslant 2h_n, \qquad v(e_n) = h_n > 0. \tag{2.6}$$

Recall that $v(z) = v(z + i0)$ for all $z \in \mathbb{R}$.

4) $u'(z) > 0$ on all (e_{n-1}^+, e_n^-) and $u(z) = \pi n$ for all $z \in g_n \neq \emptyset, n \in \mathbb{Z}$.

5) The function $k(z)$ maps a horizontal cut (a "gap") $[e_n^-, e_n^+]$ onto the vertical cut $\overline{\Gamma}_n$ and a spectral band σ_n onto the segment $[\pi(n-1), \pi n]$ for all $n \in \mathbb{Z}$.

6) The following asymptotics hold true:

$$e_n^\pm = \pi n + o(1) \qquad \text{as} \quad n \to \infty. \tag{2.7}$$

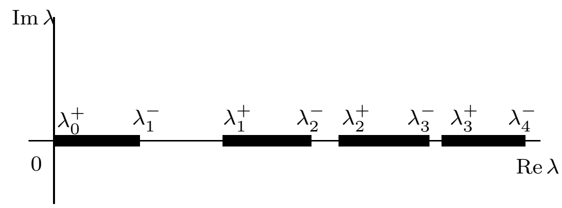

FIGURE 1. The spectral domain $\mathbb{C} \setminus \cup \mathfrak{S}_n$ and the bands $\mathfrak{S}_n = [\lambda_{n-1}^+, \lambda_n^-], n \geqslant 1$

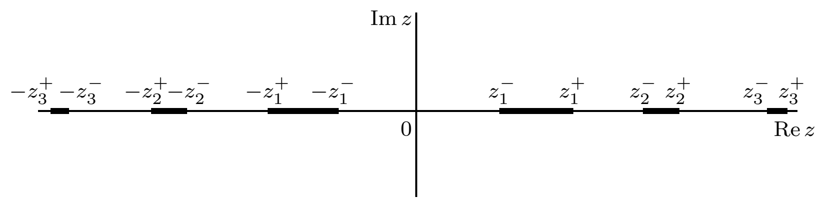

FIGURE 2. z-domain $\mathcal{Z} = \mathbb{C} \setminus \cup g_n$, where $z = \sqrt{\lambda}$ and momentum gaps $g_n = (e_n^-, e_n^+)$

7) The following identity holds true:

$$k(z) = z + \frac{1}{\pi} \int_g \frac{v(t)}{t-z} dt, \qquad \forall z \in \mathcal{Z}, \quad g = \cup g_n. \tag{2.8}$$

8) Introduce the moments

$$Q_m = \frac{1}{\pi} \int_{\mathbb{R}} t^m v(t+i0) dt < \infty, \qquad m \geqslant 0$$

and note that $Q_m = 0$ for odd $m \geqslant 1$. Then the following identities and estimate hold true:

$$Q_{2m+2} = P_m, \qquad m \geqslant -1, \tag{2.9}$$

$$\|h\|_\infty^2 \leqslant 2Q_0. \tag{2.10}$$

If $p \in \mathscr{H}_0$, then the quasimomentum $k(\cdot)$ has the asymptotics (see [K2])

$$k(z) = z - \frac{Q_0}{z} - \frac{Q_2 + o(1)}{z^3} \qquad as \quad \operatorname{Im} z \to \infty. \tag{2.11}$$

Recall the identity from [KK]. For each $n \in \mathbb{Z}$ the following identity holds true:

$$\begin{aligned} &v(z+i0) = v_n(z)(1 + Y_n(z)), \qquad \forall z \in g_n, \\ &v_n(z) = |(z-e_n^+)(z-e_n^-)|^{\frac{1}{2}}, \qquad Y_n(z) = \frac{1}{\pi} \int_{\mathbb{R} \setminus g_n} \frac{v(t)dt}{v_n(t)|t-z|}. \end{aligned} \tag{2.12}$$

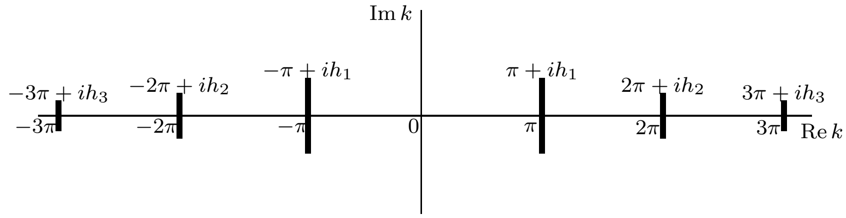

FIGURE 3. k-plane and cuts $\Gamma_n = (\pi n - ih_n, \pi n + ih_n),\ n \in \mathbb{Z}$

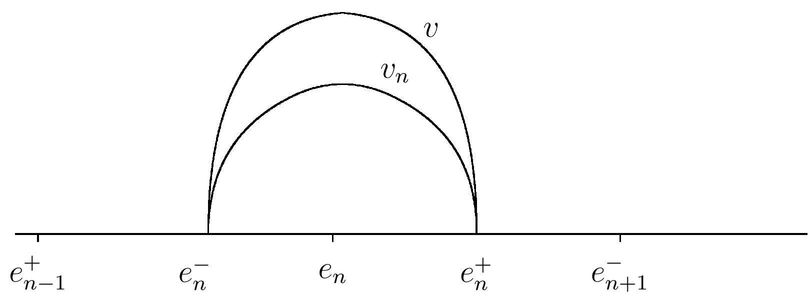

FIGURE 4. The graph of $v(z + i0),\ z \in g_n \cup \sigma_n \cup \sigma_{n+1}$ and $h_n = v(e_n + i0) > 0$

Lemma 2.1. *Let $Q_{2m} < \infty$ for some $m \geqslant 0$ and $s = \min_{n\geqslant 1} |\sigma_n|$ and $M_n = \frac{1}{\pi}\int_{g_n} v(x)dx, n \in \mathbb{Z}$. Then each function $Y_n, n \geqslant 1$, satisfies*

$$Y_n^0 := \max_{z\in g_n} Y_n(z) \leqslant \sum_{j\neq n} \frac{M_j}{s^2|n-j|^2} \leqslant \frac{Q_0}{s^2}, \qquad if \qquad m = 0, \tag{2.13}$$

$$Y_n^0 \leqslant \frac{4Q_2}{n^2 s^4}, \qquad if \qquad m \geqslant 1. \tag{2.14}$$

Proof. Using the estimate $\operatorname{dist}\{g_n, g_j\} \geqslant s|n-j|$ we obtain

$$Y_n(z) = \frac{1}{\pi}\int_{g\setminus g_n} \frac{v(t)dt}{v_n(t)|t-z|} = \sum_{j\neq n} \frac{1}{\pi}\int_{g_j} \frac{v(t)dt}{v_n(t)|t-z|} \leqslant \sum_{j\neq n} \frac{1}{\pi}\int_{g_j} \frac{v(t)dt}{s^2|n-j|^2} = \sum_{j\neq n} \frac{M_j}{s^2|n-j|^2},$$

which gives (2.13). If $m \geqslant 1$, then the above estimates and $\frac{1}{|j||n-j|} \leqslant \frac{2}{|n|}, j \neq n$ yield

$$Y_n(z) \leqslant \sum_{j\neq n} \frac{1}{\pi}\int_{g_j} \frac{t^2 v(t)dt}{s^2|n-j|^2 t^2} \leqslant \sum_{j\neq n} \frac{1}{\pi}\int_{g_j} \frac{t^2 v(t)dt}{s^4|n-j|^2 j^2} \leqslant \sum_{j\neq n} \frac{4}{n^2\pi s^4}\int_{g_j} t^2 v(t)dt = \frac{4Q_2}{n^2 s^4}.$$

■

We prove the main technical lemma of our paper.

Lemma 2.2. *i) Let $Q_{2m} < \infty$ for some $m \geqslant 0$. Then the quasimomentum has the form*

$$k(z) = z - K_{m-1}(z) + f_m(z), \qquad \forall\ z \in \mathcal{Z}, \tag{2.15}$$

where

$$f_m(z) = \frac{k_m(z)}{z^{2m}}, \qquad k_m(z) = \frac{1}{\pi}\int_g \frac{t^{2m}v(t+i0)dt}{t-z}. \tag{2.16}$$

ii) Moreover, the following estimates and asymptotics hold true:

$$|k_m(z)| \leqslant \frac{Q_{2m}}{\operatorname{dist}\{z,g\}}, \qquad \forall z \in \mathcal{Z}. \tag{2.17}$$

$$\max_{z\in g_n} |\operatorname{Im} f_m(z \pm i0)| \leqslant h_n, \qquad \max_{z\in g_n} |\operatorname{Re} f_m(z\pm i0)| \leqslant \max_{\pm} |f_m(e_n^{\pm})|, \tag{2.18}$$

$$f_m(e_n^{\pm}) = \operatorname{Re} f_m(e_n^{\pm}) = \mp \frac{|g_n|}{2}(1 + O(Y_n^0)), \tag{2.19}$$

$$\max_{z\in g_n} |f_m(z\pm i0)| = |g_n|(1 + O(Y_n^0)) \tag{2.20}$$

as $n \to \infty$, uniformly in $z \in \overline{g}_n$, where $Y_n^0 = \max_{z\in g_n} Y_n(z)$.

Proof. i) We have the simple identity

$$\frac{1}{t-z} = \frac{1}{z^{2m}}\frac{z^{2m}}{(t-z)} = \frac{1}{z^{2m}}\frac{z^{2m}-t^{2m}}{(t-z)} + \frac{1}{z^{2m}}\frac{t^{2m}}{(t-z)}.$$

Using this identity, we rewrite (2.8) in the form (here and below $v(t) = v(t+i0), t \in \mathbb{R}$)

$$k(z) - z = \frac{1}{\pi}\int_{\mathbb{R}} \frac{v(t)}{t-z}dt = \frac{1}{\pi z^{2m}}\int_{\mathbb{R}} \frac{(z^{2m}-t^{2m})v(t)}{t-z}dt + \frac{1}{\pi z^{2m}}\int_{\mathbb{R}} \frac{t^{2m}v(t)}{t-z}dt.$$

which gives (2.15), (2.16), since $Q_j = 0$ for each odd j.

ii) The identity (2.16) gives (2.17).

Using $k = z - K_{m-1} + f_m$, we obtain

$$0 \leqslant \operatorname{Im} k(z+i0) = \operatorname{Im} f_m(z+i0) = v(z+i0) \leqslant h_n, \qquad z \in g_n. \tag{2.21}$$

Now we estimate the real part $\operatorname{Re} f_m(z+i0), z \in g_n$. Using $\operatorname{Re} k(z+i0) = \pi n$ on g_n, we obtain

$$\begin{aligned} 0 = \operatorname{Re} k'(z\pm i0) = 1 - K_{m-1}'(z) + \operatorname{Re} f_m(z\pm i0)'; \\ \operatorname{Re} f_m(z\pm i0)' = -1 + K_{m-1}'(z) < -1. \end{aligned} \tag{2.22}$$

Then the function $\operatorname{Re} f_m(x \pm i0)$ is decreasing in $x \in g_n$, which yields (2.18).

We prove (2.19) for the case e_n^-. The proof for e_n^+ is similar. Using (2.16) we rewrite f_m in the form

$$f_m = f_{m1} + f_{m2}, \qquad f_{m1}(z) = \frac{1}{\pi z^{2m}}\int_{g_n} \frac{t^{2m}v(t)dt}{t-z}, \qquad f_{m2}(z) = \frac{1}{\pi z^{2m}}\int_{g\setminus g_n} \frac{t^{2m}v(t)dt}{t-z},$$

where $v(t) = v(t+i0)$. Then using (2.12) and the new variable $t = e_n^- + s$ we obtain

$$f_{m1}(e_n^-) = \frac{1}{\pi}\int_0^{|g_n|}\left(1 + \frac{s}{e_n^-}\right)^{2m} \frac{v_n(e_n^-+s)}{s}(1 + Y_n(e_n^-+s))ds$$

$$= \frac{1}{\pi}\int_0^{|g_n|}\left(1 + \frac{O(s)}{e_n^-}\right)\left|\frac{|g_n|-s}{s}\right|^{\frac{1}{2}}(1+Y_n(t))ds = I_0 + I_1,$$

where

$$I_0 = \frac{1}{\pi}\int_0^{|g_n|}\Big(1+\frac{O(s)}{e_n^-}\Big)\sqrt{\frac{|g_n|-s}{|s|}}ds, \qquad I_1 = \frac{1}{\pi}\int_0^{|g_n|}\Big(1+\frac{O(s)}{e_n^-}\Big)\sqrt{\frac{|g_n|-s}{|s|}}Y_n(t)ds \quad (2.23)$$

We have

$$\frac{1}{\pi}\int_0^{|g_n|}\sqrt{\frac{|g_n|-s}{s}}ds = \frac{|g_n|}{\pi}\int_0^1\sqrt{\frac{1-s}{s}}ds = \frac{|g_n|}{2}, \qquad (2.24)$$

and for the second term (in the case $m \geqslant 1$) we have

$$\frac{1}{\pi e_n^-}\int_0^{|g_n|}\sqrt{s(|g_n|-s)}ds = \frac{|g_n|^2}{\pi e_n^-}\int_0^1\sqrt{s(1-s)}ds = \frac{|g_n|^2}{2e_n^-}$$

which yields

$$I_0 = \frac{|g_n|}{2}\Big(1+\frac{O(|g_n|)}{n}\Big). \qquad (2.25)$$

Next, we consider I_1. Using (2.24) we have

$$I_1 = \frac{1}{\pi}\int_0^{|g_n|}\sqrt{\frac{|g_n|-s}{s}}ds O(Y_n^0) = |g_n|O(Y_n^0), \qquad Y_n^0 = \max_{t\in g_n} Y_n(t), \qquad (2.26)$$

which together with Lemma 2.1 yields (2.19). In order to study I_1 we need to consider Y_n. ■

Proof of Theorem 1.1. Identities (2.15) and (2.16) imply (1.11), which yields (1.13). Thus we have

$$k = z - K_m(z) + \frac{k_{m+1}(z)}{z^{2m+2}}, \qquad k_{m+1}(z) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{t^{2m+2}v(t)dt}{t-z}, \qquad z \in \mathcal{Z}. \qquad (2.27)$$

In order to show (1.12) we recall the well known Nevanlinna Theorem (see [Ah]).
Let μ be a Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1+x^{2m})d\mu(x) < +\infty$ for some $m \geqslant 0$. Then for each $A > 0$ the following asymptotics hold true:

$$\int_{\mathbb{R}}\frac{d\mu(t)}{t-z} = -\sum_{k=0}^{2m}\frac{q_k}{z^{k+1}} + o(\frac{1}{z^{2p+1}}) \quad \text{as} \quad |z|\to\infty, \quad y > A|x|,$$

where $q_j = \int_{\mathbb{R}} x^j d\mu(x),\ 0 \leqslant j \leqslant 2m$. Applying Nevanlinna Theorem to (2.27) and using (2.9) we obtain (1.12).

The asymptotics (2.20) and (2.17) give

$$|f_{m+1}(z)| \leqslant \frac{|\gamma_n|}{2\pi n} + b_n, \qquad b_n = \frac{O(|\gamma_n|)}{n^3} + \frac{O(1)}{n^{2m+2}} \qquad z \in \partial U_n,$$

where $U_n = \{z \in \mathcal{Z} : \mathrm{dist}\{z, g_n\} \leqslant \varepsilon\}$. This yields (1.14) since the function f_{m+1} is analytic in U_n.

The asymptotics (1.15) have been proved in (2.19). ■ .

3. Asymptotics of the fundamental solutions

We recall some known facts from Lemma 3.1 in [MO]. Define the solution y of the equation $-y''+qy=z^2y, z\neq 0$ in the form

$$\begin{aligned} y(x,z)=e^{\varkappa(x,z)}, \qquad \varkappa(x,z)=izx+\int_0^x \varkappa_*(t,z)dt,\\ \varkappa_*(x,z)=\sum_1^m \frac{\varkappa_j(x)}{(2iz)^j}+\frac{\varkappa_m(x,z)}{(2iz)^m},\\ y(0,z)=1, \qquad y'(0,z)=\varkappa'(0,z), \end{aligned} \tag{3.1}$$

for some $m\geqslant 1$. The function $\varkappa_*$ satisfies the equation:

$$(2iz)\varkappa_*+\varkappa_*'+\varkappa_*^2=p. \tag{3.2}$$

Moreover, the coefficients $\varkappa_j$ satisfy the following systems:

$$\varkappa_{j+1}=-\varkappa_j'-\sum_1^{j-1}\varkappa_{j-s}\varkappa_s, \quad j\geqslant 1, \tag{3.3}$$

where

$$\begin{aligned} \varkappa_1=p, \quad \varkappa_2=-p', \quad \varkappa_3=p''-p^2, \quad \varkappa_4=-p'''+4pp', \quad ....\\ \varkappa_j=(-1)^{j-1}p^{j-1}+\mathcal{P}_{j-3}, \quad j=1,...,m-1, \end{aligned} \tag{3.4}$$

where $\mathcal{P}_j$ is a polynomial in $p,p',p'',...,p^j$. The remainder $\varkappa_m(x,z)$ satisfies

$$\begin{aligned} \varkappa_m(0,z)=\varkappa_m(0,z)=0,\\ \varkappa_m(x,z)=O(1), \qquad \varkappa_m'(x,z)=O(z) \end{aligned} \tag{3.5}$$

as $|z|\to\infty$ uniformly in $[0,1]\times\{z\in\mathbb{C}: \ |\operatorname{Im} z|\leqslant r\}$ for any $r\geqslant 1$. Define the functions

$$\begin{aligned} \rho(z)\equiv\varkappa'(0,z)=iz+\sum_1^m\frac{\varkappa_j(0)}{(2iz)^j}\equiv\tau(z)+i\omega(z),\\ \omega(z)=\frac{\rho(z)-\rho(-z)}{2i}, \quad \tau(z)=\frac{\rho(z)+\rho(-z)}{2}, \end{aligned} \tag{3.6}$$

where

$$\omega(z)=z-\sum_0^{\frac{m-1}{2}}\frac{\varkappa_{2j+1}(0)}{(2z)^{2j+1}}, \qquad \tau(z)=\sum_1^{\frac{m}{2}}\frac{\varkappa_{2j}(0)}{(2z)^{2j}}, \tag{3.7}$$

$$\varkappa'(1,z)=\rho(z)+\frac{\varkappa_m(1,z)}{(2iz)^m}. \tag{3.8}$$

We rewrite the fundamental solutions ϑ,φ in the forms

$$\begin{aligned} \varphi(x,z)=\frac{y(x,z)-y(x,-z)}{2i\omega(z)},\\ \varphi'(x,z)=\frac{y(x,z)\varkappa'(x,z)-y(x,-z)\varkappa'(x,-z)}{2i\omega(z)}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \vartheta(x,z) &= \frac{y(x,-z)\rho(z) - y(x,z)\rho(-z)}{2i\omega(z)}, \\ \vartheta'(x,z) &= \frac{y(x,-z)\varkappa'(x,-z)\rho(z) - y(x,z)\varkappa'(x,z)\rho(-z)}{2i\omega(z)}. \end{aligned} \tag{3.10}$$

Note that in (3.1)-(3.10) we do not use the condition $E_0^+ = 0$. Recall that the set $\mathcal{Z}(\varepsilon)$ is given by $\{z \in \mathcal{Z}, \operatorname{dist}\{z, g\} > \varepsilon\}, \varepsilon > 0$.

Lemma 3.1. *Let $p \in \mathscr{H}_m$ for some $m \geqslant 0$ and let $r \geqslant 1$. Then the following asymptotics hold true:*

$$\Delta(z) = \frac{y(1,z) + y(1,-z)}{2} + O(z^{-m}) = \cos \xi_m(1,z) + O(z^{-m}), \tag{3.11}$$

$$\xi(z) := \xi_m(1,z) = z - \sum_{0 \leqslant j \leqslant \frac{m-1}{2}} (-1)^j \int_0^1 \frac{\varkappa_{2j+1}(t)}{(2z)^{2j+1}} dt, \tag{3.12}$$

$$y(1,z) = e^{i\xi(z) + O(z^{-m})}, \tag{3.13}$$

and if in addition $E_0^+ = 0$, then

$$\begin{aligned} & & k(z) = \xi(z) + O(z^{-m}), \\ \frac{(-1)^j}{2^{2j+1}} \int_0^1 \varkappa_{2j+1}(t)dt = Q_{2j}, & & \int_0^1 \varkappa_{2j}(t)dt = 0, \quad j \geqslant 0, \end{aligned} \tag{3.14}$$

as $|\operatorname{Im} z| \leqslant r$ and $|z| \to \infty$.

Proof. Let $A(z) = \varkappa'(1,z) - \rho(-z)$. Identities (3.6)-(3.7) yield

$$A(z) = 2i\omega(z) + \frac{\varkappa_m(1,z)}{(2iz)^m} = 2i\omega(z) + O(z^{-m}).$$

Then this asymptotics and (3.9), (3.10) imply

$$\Delta(z) = \frac{1}{4i\omega(z)} \Big(y(1,z)\varkappa'(1,z) - y(1,-z)\varkappa'(1,-z) + y(1,-z)\rho(z) - y(1,z)\rho(-z) \Big)$$

$$\frac{y(1,z)A(z) - y(1,-z)A(-z)}{4i\omega(z)} = \frac{y(1,z) + y(1,-z)}{2} + O(z^{-m}) = \cos i\varkappa(1,z) + O(z^{-m})$$

The function Δ is real on the real line, which gives (3.12), (3.13).

If $E_0^+ = 0$, then the identity $\Delta(z) = \cos k(z), z \in \mathcal{Z}$ and the asymptotics (3.11), (3.12) and the asymptotic estimate (2.17) imply (3.14). ■

Below, we need:

Lemma 3.2. *Let $p \in \mathscr{H}_m$ for some $m \geqslant 0$ and let $r \geqslant 1$. Then the following asymptotics hold true:*

$$\begin{aligned} \varphi(1,z) &= \frac{\sin \xi(z)}{\omega(z)} + O(z^{-m-1}), \\ \varphi'(1,z) &= \cos \xi(z) + \frac{\tau(z)}{\omega(z)} \sin \xi(z) + O(z^{-m}), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \vartheta(1,z) &= \cos\xi(z) - \frac{\tau(z)}{\omega(z)}\sin\xi(z) + O(z^{-m}), \\ \vartheta'(1,z) &= -\frac{\rho(z)\rho(-z)}{\omega(z)}\sin\xi(z) + O(z^{-m+1}), \end{aligned} \tag{3.16}$$

and

$$\beta(z) = \frac{\tau(z)}{\omega(z)}\sin\xi(z) + O(z^{-m}), \tag{3.17}$$

as $|\operatorname{Im} z| \leqslant r$ *and* $|z| \to \infty$.

Proof. Substituting the asymptotics (3.13) into the identity (3.9) we have

$$\varphi(1,z) = \frac{y(1,z) - y(1,-z)}{2i\omega(z)} = \frac{\sin\xi(z) + O(z^{-m})}{\omega(z)} = \frac{\sin\xi(z)}{\omega(z)} + O(z^{-m-1}),$$

and using additionally (3.8), (3.6) we have

$$\varphi'(1,z) = \frac{y(1,z)\varkappa'(1,z) - y(1,-z)\varkappa'(1,-z)}{2i\omega(z)} = \frac{y_m(1,z)\rho(z) - y_m(1,-z)\rho(-z) + O(z^{-m})}{2i\omega(z)},$$

which yields (3.15). The proof of the asymptotics in (3.16) is similar.

Using the asymptotics (3.15)-(3.16), we obtain

$$\beta(z) = \frac{\varphi'(1,z) - \vartheta(1,z)}{2} = \frac{\tau(z)}{\omega(z)}\sin\xi(z) + O(z^{-m}),$$

which yields (3.17). ■

We need the following identities

$$\begin{aligned} \Psi_\pm(0,z) &= 1, \qquad \Psi'_\pm(0,z) = M_\pm(z), \\ \Psi_\pm(1,z) = e^{\pm ik(z)}, &\qquad \Psi'_\pm(1,z) = e^{\pm ik(z)}M_\pm(z), \quad \forall\ z \in \mathcal{Z}. \end{aligned} \tag{3.18}$$

Proof of Theorem 1.2. Using (3.15), (3.17) and $k(z) = \xi(z) + O(z^{-m})$ (see (3.14)), we have

$$\begin{aligned} M_\pm(z) = \frac{\beta(z) \pm i\sin k(z)}{\varphi(1,z)} &= \frac{\frac{\tau(z)}{\omega(z)}\sin k(z) + O(z^{-m}) \pm i\sin k(z)}{\frac{\sin k(z)}{\omega(z)} + O(z^{-m})} \\ &= \frac{\sin k(z)\rho(\pm z) + O(z^{1-m})}{\sin k(z) + O(z^{1-m})}. \end{aligned}$$

Moreover, if $z \in \mathcal{Z}_\varepsilon$, then

$$M_\pm(z) = \rho(\pm z) + O(z^{1-m}), \tag{3.19}$$

which yields (1.21). Using (3.9), (3.10), (3.19) and (3.1), we obtain

$$\begin{aligned} \Psi_+(x,z) &= \frac{1}{2i\omega(z)}\Big[y(x,-z)\rho(z) - y(x,z)\rho(-z) + M_+(z)(y(x,z) - y(x,-z))\Big] = \\ &= \frac{1}{2i\omega(z)}\Big[y(x,z)(\rho(z) - \rho(-z)) + O(z^{1-m})(y(x,z) - y(x,-z))\Big] \\ &= y(x,z) + O(z^{-m})(y(x,z) - y(x,-z)) = y(x,z) + O(z^{-m}) = e^{i\xi(x,\pm z)} + O(z^{-m}). \end{aligned}$$

The proof for Ψ_- is similar. This yields (1.22).

The asymptotics (1.23) were proved in Lemma 3.1. ■

4. Asymptotics for the distributions

In this Section we will determine the asymptotics of the quasimomentum for the Schrödinger operator H acting in the Hilbert space $L^2(\mathbb{R})$, given by

$$Hy = -y'' + (c + p')y.$$

Here p is a 1-periodic function belonging to the real Hilbert space $\mathscr{H}_*$ given by

$$\mathscr{H}_* = \left\{ p \in L^2(0,1) : \int_0^1 p(x)dx = 0 \right\},$$

and c is a real constant. Thus, p' is a 1-periodic distribution, if $p' \in L^2(\mathbb{T})$, and then H corresponds to the Hill operator with L^2-potential. The situation considered in this paper, i.e. $p \in L^2(\mathbb{T})$, corresponds to a much more singular case.

We recall the results about the spectral properties of H from [K1]. The spectrum of H is purely absolutely continuous and consists of intervals $\mathfrak{S}_n = [E_{n-1}^+, E_n^-]$. These intervals are separated by the gaps $\gamma_n = (E_n^+, E_n^+)$ of length $|\gamma_n| \geqslant 0$. If a gap γ_n is degenerate, i.e. $|\gamma_n| = 0$, then the corresponding segments σ_n, σ_{n+1} merge. We choose the constant c in a way that $E_0^+ = 0$. All these facts are similar to the case of smooth potentials.

We can not introduce the standard fundamental solutions for the operator H, since the perturbation p' is very strong. Thus we need another representation of H. Define the unitary transformation $\mathscr{U} : L^2(\mathbb{R}, \eta^2 dx) \to L^2(\mathbb{R}, dx)$ as multiplication by η. Thus H is unitarily equivalent to

$$H_1 y = \mathscr{U}^{-1} H \mathscr{U} y = -\frac{1}{\eta^2}(\eta^2 y')' + (c - q^2)y = -y'' - 2py' + (c - p^2)y, \quad \eta = e^{\int_0^x p(t)dt}$$

acting in $L^2(\mathbb{R}, \eta^2 dx)$. This representation is more convenient, since we can introduce the fundamental solutions $\varphi_1(x,z), \vartheta_1(x,z)$ of the equation

$$-y'' - 2qy' + (c - p^2)y = z^2 y, \qquad z \in \mathbb{C}, \tag{4.1}$$

with the conditions: $\varphi_1(0,z) = \vartheta_1'(0,z) = 0, \varphi_1'(0,z) = \vartheta_1(0,z) = 1$. Define the Lyapunov function

$$\Delta(z) = \frac{\varphi_1'(1,z) + \vartheta_1(1,z)}{2}.$$

Similar to the case of smooth potentials, we introduce the quasimomentum $k(z)$ and the momentum domain $\mathcal{Z}$ and the quasimomentum domain $\mathcal{K}$ by (1.5) and (1.7). The quasimomentum $k(z)$ is a conformal mapping from $\mathcal{Z}$ onto $\mathcal{K}$, and it satisfies the standard properties (1.6) and (2.4)-(2.12). To characterize the quasimomentum $k(z)$ further, we recall the results from [K1]: The quasimomentum has the form

$$k(z) = z - k_0(z), \qquad k_0(z) = \frac{1}{\pi} \int_g \frac{v(t)dt}{t - z}, \qquad \forall\ z \in \mathcal{Z}, \tag{4.2}$$

and for any $A > 0$, the following asymptotics hold true

$$k(z) = z - \frac{P_{-1} + o(1)}{z} \qquad as\ |z| \to \infty, \quad y > A|x|. \tag{4.3}$$

Here, the coefficient P_{-1} has the form

$$P_{-1}=\frac{\|q\|^2}{2}=\frac{1}{\pi}\int_{\mathbb{R}}v(t+i0)dt=\frac{1}{2\pi}\iint_{\mathbb{C}}|k'(z)-1|^2dxdy,\qquad z=x+iy,\tag{4.4}$$

where $q\in\mathscr{H}_*$ is a solution of the Riccati equation

$$p'=q'(x)+q(x)^2-\|q\|^2.\tag{4.5}$$

Recall that the mapping $p\to q$ acting from $\mathscr{H}_*$ into $\mathscr{H}_*$ is a real analytic isomorphism onto itself. Thus for each $p\in\mathscr{H}_*$ there exists a unique solution $q\in\mathscr{H}_*$ of the equation (4.5).

Define the sequence $S_n, n\geqslant 1$ by

$$S_n(r)=\sum_{j\in\mathbb{Z}}\frac{M_j}{|n-j|_1},\qquad |j|_1=\begin{cases}s|j|, & if\ j\neq 0\\ \frac{r}{2}, & if\ j=0\end{cases},\quad M_j=\frac{1}{\pi}\int_{g_j}v(t+i0)dt.\tag{4.6}$$

Note that simple estimates yield

$$\sum_{n\geqslant 1}S_n^a(r)\leqslant\sum_{n\geqslant 1}\sum_{j\in\mathbb{Z}}\frac{M_j}{|n-j|_1^a}\biggl(\sum_{j'\in\mathbb{Z}}M_{j'}\biggr)^{a-1}\leqslant Q_0^a\biggl(\frac{4^a}{r^{2a}}+\frac{2}{s^a}C_a\biggr),$$

where $C_a=\sum_{n\geqslant 1}\frac{1}{|j|^a}$. This yields

$$\sum_{n\geqslant 1}S_n^a(r)\leqslant 4Q_0^a\biggl(\frac{1}{r^2}+\frac{1}{s^2}\biggr),\qquad if\quad a>1.\tag{4.7}$$

Define the domains

$$V_n(r)=\biggl\{z\in\mathbb{C}:|\operatorname{Im}z|\leqslant r,\quad \frac{e_n^-+e_{n-1}^+}{2}<\operatorname{Re}z<\frac{e_n^++e_{n+1}^-}{2}\biggr\}\setminus U_n,\ r>\pi,\tag{4.8}$$

$$U_n=\biggl\{z\in\mathcal{Z}:\operatorname{dist}\{z,g_n\}\leqslant\varepsilon\biggr\}.\tag{4.9}$$

Theorem 4.1. *Let $H=-\frac{d^2}{dx^2}+(c+p')$ for some $p\in\mathscr{H}_0$, and let $r\geqslant\pi$. Then for each $\varepsilon>0$ small enough, $k_0(z)$ satisfies*

$$\max_{z\in g_n}|k_0(z\pm i0)|=|g_n|(1+O(Y_n^0))\tag{4.10}$$

$$\max_{\operatorname{dist}\{z,g_n\}=\varepsilon}|k_0(z)|\leqslant S_n(\varepsilon),\tag{4.11}$$

$$\max_{z\in\partial V_n\setminus\partial U_n}|k_0(z)|\leqslant S_n(1),\tag{4.12}$$

$$\max_{z\in U_n}|k_0(z\pm i0)|=|g_n|(1+O(Y_n^0)),\tag{4.13}$$

$$\max_{z\in V_n}|k_0(z)|\leqslant S_n(\varepsilon)+S_n(s),\tag{4.14}$$

Proof. The asymptotics (4.10) follows from (2.20).

In order to show (4.11), we write the simple estimate of k_0 in the form:

$$|k_0(z)|\leqslant\frac{1}{\pi}\int_g\frac{v(t)dt}{|t-z|}=\sum_{j\in\mathbb{Z}}\frac{1}{\pi}\int_{g_j}\frac{v(t)dt}{|t-z|}.$$

Consider the summands separately in sum. We obtain estimates for $z \in U_n$:

$$\frac{1}{\pi}\int_{g_n}\frac{v(t)dt}{|t-z|} \leqslant \frac{1}{\pi\varepsilon}\int_{g_n} v(t)dt = \frac{M_n}{\varepsilon}, \qquad n=j;$$

$$\frac{1}{\pi}\int_{g_j}\frac{v(t)dt}{|t-z|} \leqslant \frac{1}{\pi s}\int_{g_j}\frac{v(t)dt}{|n-j|} = \frac{M_j}{s|n-j|}, \qquad n\neq j.$$

Summing these estimates we obtain (4.11). The proof of (4.12) is similar.

The function k_0 is analytic in the domain U_n and satisfies the estimates (4.10), (4.11) on the boundary, which yields (4.13).

The function k_0 is analytic in the domain V_n and satisfies the estimates (4.11), (4.12) on the boundary, which yields (4.14). ■

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References

[Ah] Akhiezer, N. The classical moment problem and some related questions in analysis. Hafner Publishing Co., New York 1965.

[CK] Chelkak, D.; Korotyaev, E. Spectral estimates for Schrödinger operators with periodic matrix potentials on the real line. Int. Math. Res. Not. 2006, Art. ID 60314, 41 pp.

[C] Cuccagna, S. Dispersion for Schrdinger equation with periodic potential in 1D. Comm. Partial Differential Equations 33 (2008), no. 10-12, 2064-2095.

[F] Firsova, N. A direct and inverse scattering problem for a one-dimensional perturbed Hill operator. Mat. Sb. 130(172) (1986), no. 3, 349–385.

[F1] Firsova, N. Rieman surface of quasiimpulse and the theory of scattering for disturbed Hill operator, Zap. Nauchn. Sem. LOMI, 51 (1975), 183-196.

[F2] Firsova, N. Levinson formula for perturbed Hill operator, Teoret. Mat. Fiz., 62:2 (1985), 196-209.

[GT] Garnett J., Trubowitz E.: Gaps and bands of one dimentional periodic Schrodinger operator. Comment. Math. Helv. 59(1984), 258–312.

[GWH] Goodman, R. H.; Weinstein, M. I.; Holmes, P. J. Nonlinear propagation of light in one-dimensional periodic structures. J. Nonlinear Sci. 11 (2001), no. 2, 123-168.

[KK] Kargaev, P.; Korotyaev, E. Effective masses and conformal mappings. Comm. Math. Phys. 169 (1995), no. 3, 597–625.

[KK1] Kargaev, P.; Korotyaev, E. Inverse Problem for the Hill Operator, the Direct Approach. Invent. Math., 129 (1997), no. 3, 567–593.

[K1] Korotyaev, E. Characterization of the spectrum of Schrödinger operators with periodic distributions. Int. Math. Res. Not. 2003, no. 37, 2019–2031.

[K2] Korotyaev E. Estimates for the Hill operator.I, Journal Diff. Eq. 162(2000), 1–26.

[K3] Korotyaev E. Estimate for the Hill operator.II, J. Differential Equations 223 (2006), no. 2, 229–260.

[K4] Korotyaev, E. Inverse problem and the trace formula for the Hill operator. II Math. Z. 231 (1999), no. 2, 345–368.

[KP] Korotyaev, E.; Pushnitski, A. On the high-energy asymptotics of the integrated density of states. Bull. London Math. Soc. 35 (2003), no. 6, 770–776.

[MO] Marchenko, V.; Ostrovski I. A characterization of the spectrum of the Hill operator. Math. USSR Sbornik 26(1975), 493–554.

[MM] McKean H., P. van Moerbeke. The spectrum of Hill's equation. Invent. Math. 30(1975), 217–274.

[PS1] Parnovski, L.; Shterenberg, R. Complete asymptotic expansion of the integrated density of states of multidimensional almost-periodic Schrödinger operators, preprint, 2010, arXiv:1004.2939.

[PS2] Parnovski, L.; Shterenberg, R. Asymptotic expansion of the integrated density of states of a two-dimensional periodic Schödinger operator. Invent. Math. 176 (2009), no. 2, 275-323.

[RS] Reed, M.; Simon, B. Methods of modern mathematical physics. IV. Analysis of operators, Academic Press, New York-London, 1978.

[SS] Shenk, D.; Shubin, M. A. Asymptotic expansion of state density and the spectral function of the Hill operator. (Russian) Mat. Sbornik. 128(1985), no. 4, 474-491.

[T] Titchmarsh, E. Eigenfunction expansions associated with second-order differential equations 2, Clarendon Press, Oxford, 1958.

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